Linking the Quark Meson Model with QCD at High Temperature

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Abstract

We model the transition of a system of quarks and gluons at high energies to a system of quarks and mesons at low energies in a consistent renormalization group approach. Flow equations interpolate between the physics of the high-temperature degrees of freedom and the low-temperature dynamics at a scale of 1 GeV. We also discuss the dependence of the equation of state on baryon density and compare our results with recent lattice gauge simulations.

The equation of state of QCD in extreme conditions of high temperature and/or high baryon density is at the focus of interest in heavy-ion collisions and astrophysics, i.e. neutron stars and supernovae. The current heavy-ion experiments at SPS (CERN) and RHIC (BNL) have challenged theoretical work on strong interactions at finite temperatures, due to the possible observation of new phases of matter. In particular, one expects a phase transition which separates a low-temperature region of hadrons from a high-temperature region of quarks and gluons, where both confinement is absent and chiral symmetry is restored. During the last years various direct numerical lattice simulations of QCD at finite temperature and finite baryon density have been performed [1–3]. These computations are still limited to rather small lattices, therefore an alternative approach using an effective field theory is useful. Previously, we have discussed a linear σ-model with two massless quark flavours [5,6] which is treated with the renormalization group to capture the dynamics of the long-range fluctuations near the critical point correctly. In particular, this method includes dynamical chiral symmetry breaking as the dominating feature of QCD at low energies. The low-energy model lacks the gauge degrees of freedom, therefore it cannot be linked with lattice calculations for $T \gtrsim 160$ MeV. As a first step, we propose to extend the flow equations of the linear σ-model by a flow equation for free massless quarks and gluons at high scale.

Combining these two systems within a single renormalization group flow which integrates successively over both high and low energy excitations one obtains an adequate equation of state of strong interaction over a large range of temperatures and densities, as a comparison of our results with recent lattice simulations shows.

The $SU(2)_L \otimes SU(2)_R$ invariant linear σ-model with constituent quarks q and chiral mesons $\Phi = (\sigma, \vec{\pi})$ is an effective model for dynamical spontaneous chiral symmetry breaking at intermediate scales of $0 \lesssim k \lesssim \Lambda_M$. It is given by the Euclidean Lagrangian density

$$\mathcal{L}_{L\sigma M} = \bar{q}\partial\!\!\!/q + g\bar{q}(\sigma + i\tau\pi\gamma_5)q + \tfrac{1}{2}(\partial_\mu\Phi)^2 + U(\Phi^2)\,.$$

The grand canonical partition function Z for the quantum theory has the path integral representation

$$Z = \mathrm{Tr}\, e^{-\beta(\mathrm{H}-\mu\mathrm{N})} \tag{1}$$
$$= \int D\bar{q} \int Dq \int D\Phi \exp\left(-\int_0^\beta dt_E \int d^3 x_E \left(\mathcal{L}_{L\sigma M} - \mu q^\dagger q\right)\right),$$

where periodic and anti-periodic boundary conditions apply for bosons and fermions, respectively. A Gaussian approximation to the path integral followed by a Legendre transformation yields the 1-loop effective action

$$\Gamma[\Phi, \bar{q}, q] = S^{\mathrm{uv}}[\Phi, \bar{q}, q] - \mathrm{Trlog}\left(\frac{\delta^2 \mathrm{S}[\Phi, \bar{q}, q]}{\delta\bar{q}(x)\delta q(y)}\right) + \tfrac{1}{2}\mathrm{Trlog}\left(\frac{\delta^2 \mathrm{S}[\Phi, \bar{q}, q]}{\delta\Phi^i(x)\delta\Phi^j(y)}\right)\,. \tag{2}$$

Here, the boundary conditions of the functional integral appear in the momentum traces and we neglect contributions from mixed quark meson loops. We consider the effective action Γ in a local potential approximation (LPA), which represents the lowest order in the derivative expansion and incorporates fermionic as well as bosonic contributions to the grand canonical potential density Ω which is related to the pressure of the system by

$$\Omega(T, \mu) = -p(T, \mu)\,. \tag{3}$$

To derive RG flow equations, we use Schwinger's proper-time method to regularize the respective logarithms. A straightforward partial evaluation of the trace in momentum space and a transformation of the expression to proper

time form[1] yields

$$\Omega = \Omega^{\text{uv}} + \frac{1}{2} \int \frac{d\tau}{\tau} T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \left(4N_c N_f \mathrm{e}^{-\tau\left((\nu_n+i\mu)^2+\vec{p}^2+M_q^2\right)} \right. \tag{4}$$
$$\left. -3\mathrm{e}^{-\tau\left(\omega_n^2+\vec{p}^2+M_\pi^2\right)} - \mathrm{e}^{-\tau\left(\omega_n^2+\vec{p}^2+M_\sigma^2\right)} \right) f(\tau k^2) .$$

where the Matsubara frequencies take the values $\omega_n = 2n\pi T$ for bosons and $\nu_n = (2n+1)\pi T$ for fermions, respectively. The effective masses are defined as

$$M_q^2 = g^2\Phi^2 , \quad M_\pi^2 = 2\frac{\delta\Omega}{\delta\Phi^2} , \quad M_\sigma^2 = 2\frac{\delta\Omega}{\delta\Phi^2} + 4\,\Phi^2\frac{\delta^2\Omega}{(\delta\Phi^2)^2} .$$

A general set of smooth cutoff functions $f(\tau k^2)$, parametrized by a real positive number a fulfills the required regularization conditions, cf. ref. [5,6,9]

$$f^{(a)}(\tau k^2) \equiv \frac{\Gamma(a+1,\tau k^2)}{\Gamma(a+1)} . \tag{5}$$

In previous works [5,6,9], cutoff functions with integer a have been applied to various problems. However, in the case of a thermal system half-integer cutoff functions are better suited, since they effectively regularize only the spatial momentum integral. The relevant derivative of the cutoff function with respect to the IR cutoff scale

$$k\frac{\partial f^{(a)}(\tau k^2)}{\partial k} = -\frac{2}{\Gamma(a+1)}(\tau k^2)^{a+1}\mathrm{e}^{-\tau k^2}$$

involves a half-integer power of the proper time variable τ, which combines with the half integer power from the integration over the spatial momentum coordinates. With $a = 3/2$ the proper time integral gives an *analytic* RG improved flow equation for the grand canonical potential Ω

$$k\frac{\partial\Omega_{L\sigma M}(\Phi^2,k)}{\partial k} = \frac{k^5}{6\pi^2}\left(\frac{3}{E_\pi}\left(\frac{1}{2}+n_B(E_\pi)\right) + \frac{1}{E_\sigma}\left(\frac{1}{2}+n_B(E_\sigma)\right) \right.$$
$$\left. -\frac{2N_c N_f}{E_q}\left(1-n_F(E_q)-\bar{n}_F(E_q)\right) \right) . \tag{6}$$

The appearing effective energies are defined by

$$E_i = \sqrt{k^2 + M_i^2} , \quad i \in \{\pi,\sigma,q\} ,$$

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[1] The transformation of the apparently complex expression to proper time form is possible, since all imaginary parts cancel each other in the Matsubara sum.

and the occupation numbers have the usual form

$$n_B(E) = \frac{1}{e^{E/T} - 1} \ , \quad n_F(E) = \frac{1}{e^{(E-\mu)/T} + 1} \ , \quad \bar{n}_F(E) = \frac{1}{e^{(E+\mu)/T} + 1} \ .$$

Note the exceptionally simple form of this evolution equation. One can immediately read off the contributions from the vacuum and thermal dynamics of the respective particles appearing with their proper degeneracy factors. Especially, compared to previous vacuum equations [9] energy denominators appear instead of the usual propagator terms, which is natural for a thermal equation because of the broken Poincaré invariance. The analytic form of these equations allows a numerical solution, especially in the case of finite temperature and density where the computational effort rises considerably.

The thermodynamics and phase structure of the chiral system has been discussed in [5–8]. One obtains a good description of the second order chiral phase transition at $T_c \approx 160$ MeV and also a qualitative picture of the first order phase transition at high densities. However, the model lacks the correct high-temperature behaviour since it does not contain the appropriate degrees of freedom in this regime.

The fundamental degrees of freedom at high energies are quarks and gluons of QCD, described by the Euclidean Lagrangian density

$$\mathcal{L}_{QCD} = \bar{q} \left(\slashed{D} + m_c \right) q + \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \ , \tag{7}$$

where m_c is the respective quark mass matrix. Due to asymptotic freedom quarks and gluons decouple at asymptotically high energies and the partition function takes a similar form as Eq. (1) with chiral mesons replaced by gluon fields. Lattice gauge simulations, cf. ref. [1,2], suggest that even at moderate temperatures $T \gtrsim 200$ MeV the equation of state of the interacting system behaves almost like a free gas. Therefore in a first attempt to link the quark meson model with QCD we neglect the full gauge interactions and approximate the high-energy part of QCD at scales $k > \Lambda_M$ by a free gas of massless quarks and gluons. The integration of the thermal fluctuations within a common RG flow using the same smooth infrared cutoff function allows a smooth transition of the respective degrees of freedom. The matching depends on the chosen cutoff function. A smooth cutoff function has the same effect as a self-interaction of the gluons and provides an effective screening for the gluonic degrees of freedom. In order to match the two systems independently of the matching scale one would have to add the relevant quark-gluon dynamics. In this paper we show that, however, already the simple addition of the free quark gluon flow with the meson flow gives reasonable results for certain quantities as the equation of state. In further work we plan to apply the evolution of the gluon system [13–15,17,18] to reduce the dependence on the matching scale.

We neglect the current quark masses for the two light quark flavors, then the

equation for the high-energy part has a very compact form in our simplified approximation:

$$k\frac{\partial \Omega_{QCD}(k)}{\partial k} = \frac{k^4}{6\pi^2}\left(N_g(\frac{1}{2} + n_B(k)) - 2N_cN_f\left(1 - n_F(k) - \bar{n}_F(k)\right)\right)$$

$$= k^4 \sum_{i\in\{g,q,\bar{q}\}} \frac{N_i}{6\pi^2}\left(\frac{s_i}{2} + n_i(k)\right) . \tag{8}$$

The sum runs over gluons, quarks and antiquarks including the appropriate occupation numbers with degeneracy factors and sign factors $N_g = 16$, $s_g = +1$ for the bosons and $N_q = N_{\bar{q}} = 2N_cN_f = 12$, $s_q = s_{\bar{q}} = -1$, for the fermions and antifermions, respectively. The potential of the total system is obtained by integrating the quark-gluon evolution equation from $k = \Lambda_\infty$ to $k = \Lambda_M$ and the quark-meson evolution equation from $k = \Lambda_M$ to $k = 0$, i.e. the IR-cutoff parameter of the quark-gluon potential is identical to the UV-cutoff parameter of the quark-meson model. We can integrate the simplified QCD flow equation analytically and get

$$\Omega_{QCD}(\Lambda_\infty, \Lambda_M, T) = \sum_i s_i \frac{N_i}{6\pi^2}\left[\frac{k^4}{8} - k^3 T Li_1\left(s_i n_i^\infty(k)\right)\right. \tag{9}$$

$$\left. -3k^2T^2 Li_2\left(s_i n_i^\infty(k)\right) - 6kT^3 Li_3\left(s_i n_i^\infty(k)\right) - 6T^4 Li_4\left(s_i n_i^\infty(k)\right)\right]_{\Lambda_\infty}^{\Lambda_M} .$$

The polylogarithmic functions $Li_n(x)$ are defined by

$$Li_1(x) = -\ln(1-x) , \quad Li_n(x) = \int_0^x \frac{Li_{n-1}(z)}{z}dz$$

and the functions $n_i^\infty(k)$ denote the classical Maxwell-Boltzmann distribution functions of the respective particles

$$n_g^\infty(k) = e^{-\frac{k}{T}} , \quad n_q^\infty(k) = e^{-\frac{k-\mu}{T}} , \quad n_{\bar{q}}^\infty(k) = e^{-\frac{k+\mu}{T}} .$$

We can evaluate eq. (9) in the limit $\Lambda_\infty \to \infty$ by making use of the behaviour of the polylogarithms in the vicinity of zero

$$\Omega_{QCD}(\Lambda_\infty, \Lambda_M, T) = \sum_i s_i \frac{N_i}{6\pi^2}\left(-\frac{\Lambda_\infty^4}{8} + \frac{\Lambda_M^4}{8} - \Lambda_M^3 T Li_1(s_i n_i^\infty(\Lambda_M))\right. \tag{10}$$

$$\left. -3\Lambda_M^2 T^2 Li_2(s_i n_i^\infty(\Lambda_M)) - 6\Lambda_M T^3 Li_3(s_i n_i^\infty(\Lambda_M)) - 6T^4 Li_4(s_i n_i^\infty(\Lambda_M))\right) .$$

Here we kept the divergent part proportional to Λ_∞^4, so the potential still needs to be regularized. In the limit of zero temperature the vacuum part can be identified, since it is only a function of Λ_M and Λ_∞

$$\Omega_{QCD}^{vac}(\Lambda_\infty, \Lambda_M, T = 0) = \sum_i s_i \frac{N_i}{6\pi^2}\left(\frac{\Lambda_M^4}{8} - \frac{\Lambda_\infty^4}{8}\right) . \tag{11}$$
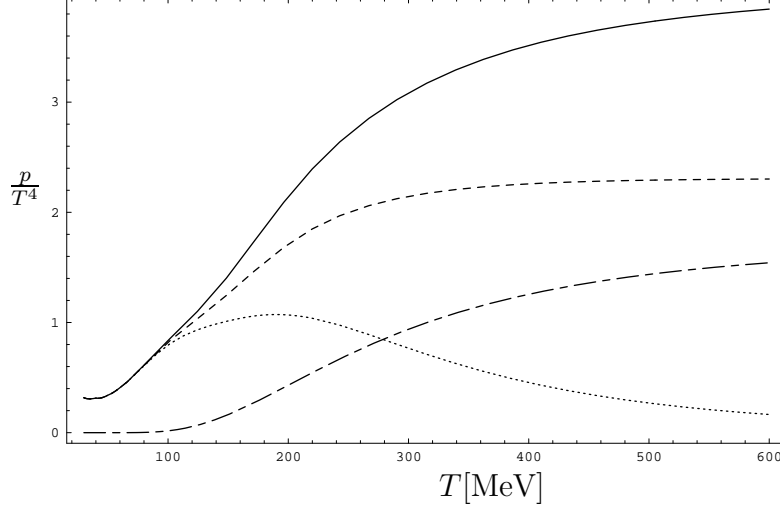
Fig. 1. Different contributions to $\frac{p}{T^4}$ arising from the mesons and quarks (constituent quarks and free quarks) (dashed line) and the gluons (dash-dotted line). The dotted line shows the scaled pressure function of the linear σ-model with two quark flavours which contains the cutoff Λ_M.

To obtain the potential $\Omega_{L\sigma M}$ of the chiral low-energy part, we integrate the corresponding flow equation from $k = \Lambda_M$ to k=0. As initial values we choose $\Lambda_M = 1.0$ GeV, $U(\Phi^2) = \frac{1}{2}m_M^2\Phi^2 + \frac{1}{4}\lambda_M\Phi^4$ with $\lambda_M = 37$, $m_M = 320$ MeV and the quark-meson coupling $g = 3.45$. These initial parameters are adjusted to get the value of the pion decay constant $< \sigma >= f_\pi = 88$ MeV at k=0 in the chiral limit. In principle, the starting parameters should result from the integration over the QCD degrees of freedom. This may be possible including quark interactions from gluons at higher scales.

The full potential reads:

$$\Omega_{total}(\Lambda_\infty, k = 0, T) = \Omega_{QCD}(\Lambda_\infty, \Lambda_M, T) + \Omega_{L\sigma M}(\Lambda_M, k = 0, T). \qquad (12)$$

The finite-temperature part relevant for the pressure does not depend on the vacuum energy. Therefore in the regularized potential the UV-divergent part drops out as it must

$$p(T) = -\Omega_{total}(\Lambda_\infty, k = 0, T) + \Omega_{total}(\Lambda_\infty, k = 0, T = 0). \qquad (13)$$

In the limit of infinite temperature $T \to \infty$ the ratio of p/T^4 yields the Stefan-Boltzmann limit for a massless quark-gluon gas, because the region $k < \Lambda_M$ has vanishing weight

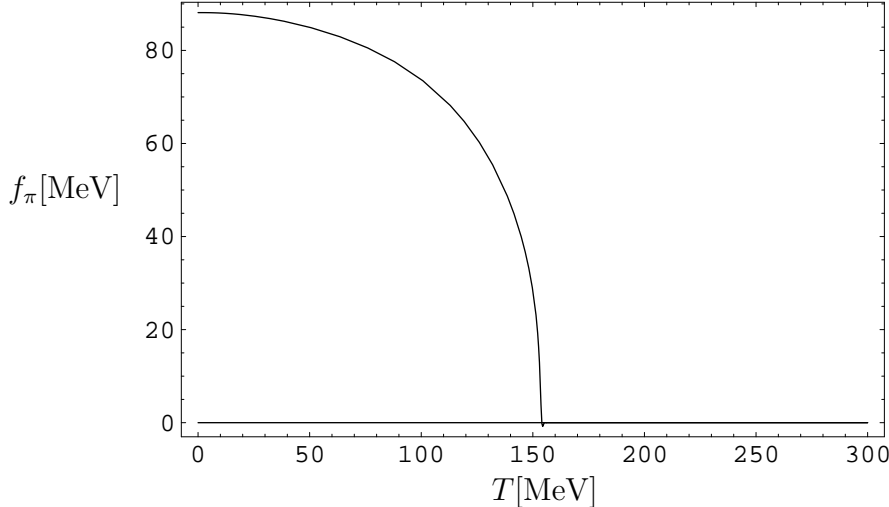$$\frac{p(T)}{T^4} \to \left(N_g + \frac{7}{8}\left(N_q + N_{\bar{q}} \right) \right) \frac{\pi^2}{90}. \qquad (14)$$

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Fig. 2. .The pion decay constant as a function of temperature for vanishing chemical potential.

For small temperatures $T \to 0$ we find a massless gas of pions with

$$\frac{p(T)}{T^4} \to 3\frac{\pi^2}{90} \, . \tag{15}$$

In fig. 1 we show the different contributions to p/T^4 arising from the integration over the flow of the mesons and all quark contributions (including both the constituent quarks with cutoff Λ_M and free quarks without cutoff)(dashed line) and the integration over the gluons (dash-dotted line). Furthermore we plot the scaled pressure of the linear σ-model with cutoff Λ_M (dotted line) described by eq. (6).

The comparison indicates clearly that the quark-meson model with cutoff Λ_M alone becomes unreliable for a description of the equation of state at temperatures $T > 150$ MeV. It definitely has to be linked with quark and gluon degrees of freedom at higher temperatures. The linear sigma model is adequate at low temperature where massless pions dominate the pressure and the quarks are massive due to spontaneous breaking of the chiral symmetry. Gluons contribute to the pressure only at rather high temperatures since the smooth infrared cutoff around the scale Λ_M introduces a gluon mass and gives deviations between our model and the free gluon gas. The flow equation with the successive integration over different degrees of freedom avoids a first-order transition which typically arises in simplified models with two components like the bag model combined with a pion gas [16].

The critical behaviour of the system is described by the pion decay constant which is shown as a function of temperature for vanishing chemical potential in fig. 2. The figure shows that the pion decay constant goes to zero for $T_c = 154$ MeV continuously. For temperatures below T_c the quarks are massive
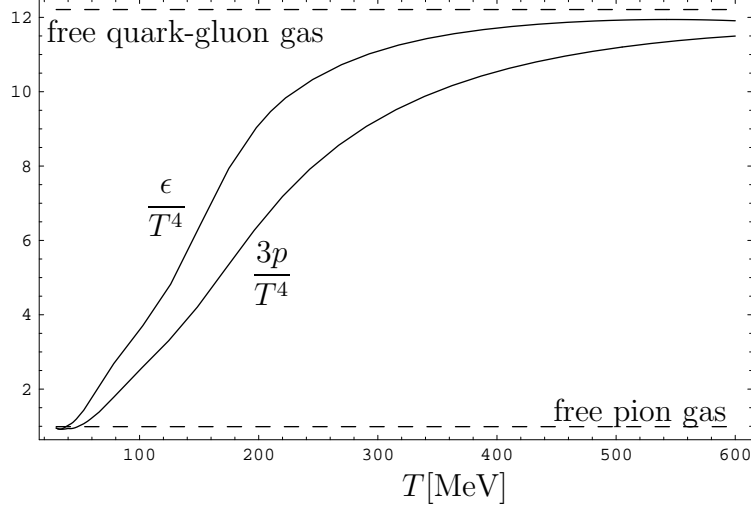
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Fig. 3. The scaled pressure density multiplied by a factor of three compared to the scaled energy density as a function of temperature. The upper and lower Stefan-Boltzmann limits corresponding to a free quark-gluon gas and a pion gas are drawn as dashed lines.

and the pions are massless, so the system is in the chiral broken phase. For temperatures above T_c the system is in the chiral symmetric phase, where the quarks are massless and the pions are massive. So one obtains a second order chiral phase transition at $T = T_c$, which is about 20 MeV smaller than the result of lattice calculations [2]. Note that the pion decay constant as a function of temperature for the chiral low energy system does not differ from the function of the total system, because the pion decay constant is zero in the high-energy part of our model.

In fig. 3 we compare the curve of the scaled energy $\frac{\varepsilon}{T^4}$ with three times the pressure $\frac{3p}{T^4}$ as a function of temperature. Both quantities $\frac{\varepsilon}{T^4}, \frac{3p}{T^4}$ gradually increase across the second order phase transition. The scaled pressure density reaches about 95% of the Stefan-Boltzmann limit at $T \approx 4\,T_c$. Respective lattice calculations [2] reach about 85% of the Stefan-Boltzmann limit at $T \approx 4\,T_c$. The energy density becomes more rapidly asymptotic, cf. fig. 3. Thereby, the "interaction measure" $\varepsilon - 3p$ has a peak at $T \approx 1.27\,T_c \approx 196\,\mathrm{MeV}$, whereas the respective maximum of lattice calculations of (2+1)-flavour QCD [3] is located at approximately $T \approx 1.1\,T_c \approx 189\,\mathrm{MeV}$. The "interaction measure" $\varepsilon - 3p$ has more subtle features at finite chemical potential. For instance we find a low temperature maximum in $\varepsilon - 3p$ at finite chemical potential which does not agree with finite density lattice calculations. This low temperature maximum arises from the fact that we have constituent quarks in our model and no baryons for temperatures below T_c. Lattice calculations of the same quantity go rapidly to zero for temperatures below T_c which may be an effect of confinement and/or of the limited grid sizes and the high masses of the nucleons.
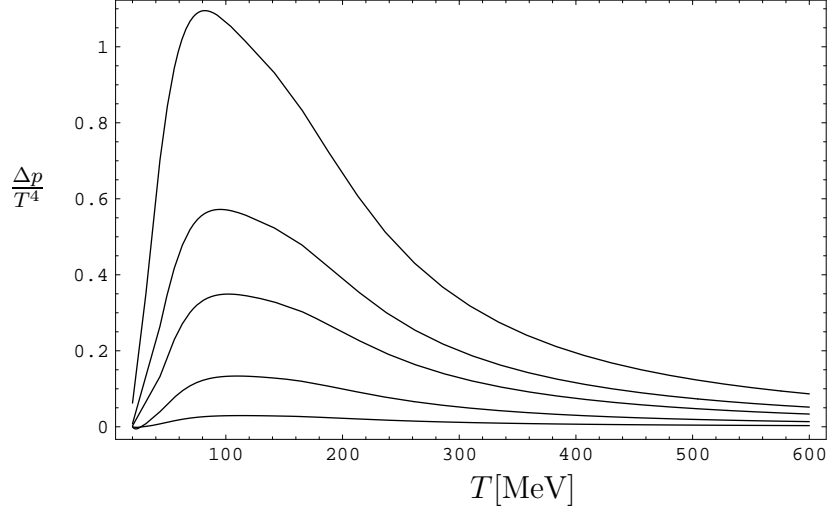
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Fig. 4. The scaled pressure difference $\frac{\Delta p}{T^4} = \frac{p(T,\mu)-p(T,\mu=0)}{T^4}$ is shown as a funtion of temperature for $\mu_B = 100, 210, 330, 410, 530$ MeV (from bottom to top).

To study the effects of a finite baryon chemical potential on the pressure it is useful to consider the difference Δp of the pressure of the baryonic system to the nonbaryonic system at the same temperature:

$$\Delta p = (p(T,\mu) - p(T,\mu = 0)) . \tag{16}$$

Small chemical potentials compared to the baryon mass suppress the baryonic pressure at low temperatures. Our calculation shows a suppression of $\Delta p/T^4$ in agreement with the low-temperature expansion for a gas of nonrelativistic and non-interacting constituent quarks with masses $m > \mu = \mu_B/3$, which reads

$$\frac{\Delta p}{T^4} = \frac{g}{2\sqrt{2}\pi^{\frac{3}{2}}} \left(\frac{m}{T}\right)^{\frac{3}{2}} \left(e^{-\frac{m-\mu}{T}} - e^{-\frac{m}{T}}\right) . \tag{17}$$

Here g denotes the degeneracy factor of the fermions. In fig. 4 one can see the scaled pressure $\frac{\Delta p}{T^4}$ of the quarks for different chemical potentials $\mu_B = 100$ MeV to $\mu_B = 530$ MeV. We have chosen these values identical to the ones in the lattice calculation in ref. [3], where the equation of state includes heavy strange quarks. At high temperatures the scaled pressure difference decreases according to

$$\frac{\Delta p}{T^4} = 4N_c N_f \left(\frac{\mu^2}{24T^2} + \frac{\mu^4}{48\pi^2 T^4}\right) . \tag{18}$$

This behaviour is also reproduced by our calculation, cf. fig. 4.

Linking the quark-meson model with QCD degrees of freedom we find that the low-energy sigma model can be extended to high temperature $T > 150$ MeV. The combination of different degrees of freedom within the renormalization group flow equations opens the window for a consistent description of QCD both at low and high scales which is needed in the equation of state. Already the simplified model we constructed shows good agreement with lattice simulations. As indicated above, the low-momentum behaviour of the model has to be

improved to get better agreement with the physics for temperatures below T_c. Furthermore the matching between the quark-gluon and quark-meson model is too simple and should be extended by including the relevant quark-gluon dynamics on the high-energy side. Both the effective four-fermion coupling and gluon condensation depend on the matching scale. A scale independent transition may be achieved if one can connect the chiral symmetry dynamics with gluon condensation.

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